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INTERGENERATIONAL EQUITY AND EFFICIENT ALLOCATION OF EXHAUSTIBLE RESOURCES*

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1. INTRODUCTION

A problem of long-standing interest in intertemporal welfare economics is whether an appropriate concept of intergenerational equity is compatible with efficient allocation of resources.

In a model with exhaustible resources, Solow [1974] has interpreted intergenerational equity to mean equal consumption per capita at each date. The problem is to find a path (given arbitrary initial conditions) which is equitable in this sense, and is also intertemporally efficient. If there exists such a path, then clearly there is no conflict between equity and efficiency. Furthermore, such a path has an additional feature, namely it is "maximin" — the rule of distributive justice, proposed by Rawls [1971].

A necessary condition for the existence of an efficient equitable path is that there is some path which can maintain a positive consumption level. Solow [1974] confines his analysis to the case where the production function is Cobb-Douglas, and capital does not depreciate. In this case, a necessary and sufficient condition for the existence of a path which can maintain a positive consumption level is that the share of capital in current output exceeds the share of the exhaustible resource in current output. With this additional condition, Solow proves the existence of an efficient equitable path.

The first purpose of this paper is to solve the existence problem posed by Solow, for a general production function on capital, labor, and an exhaustible resource. We *assume* the existence of an equitable path with positive consumption. (The general necessary and sufficient technological conditions, under which this assumption is satisfied, have been obtained in Cass and Mitra [1979].) We then prove (in Theorem 1) the existence of an efficient equitable path, when the exhaustible resource is "important" in production (in the sense of assumptions (A.3) and (A.4)). When the resource is not "important" (particularly in the sense of (A.3)) the result may not be true. We provide an example to demonstrate this (see Example 1). We remark (see Remark 3) that a weaker version of (A.4) which only requires the assumption to be valid along certain paths (assumption (A.4′)) is enough to establish the result. An example (Example 2) is provided where (A.3) is satisfied, (A.4) is violated, there exists an efficient and equitable path from

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every positive initial stock and (A.4′) is satisfied. This shows that (A.4′) is genuinely weaker than (A.4). We have, however, retained (A.4), in the main body of our results, which being a general assumption on technology rather than on specific paths, is possibly easier to verify and less awkward. We also recognize the possibility that the assumptions used to prove the theorem (in particular (A.4′)) may not be the weakest possible, although they seem essential to the method of proof used to obtain the existence result. We do not have an example to show whether or not (A.4′) is necessary. Whether an alternative method of proof can be devised to establish the result under weaker assumptions remains an open question.

The second purpose of this paper is to provide a price-characterization for efficient equitable (or maximin) paths. (Such characterizations are abundant in the theory of optimal economic growth, à la Ramsey.) A motivation for their study is to examine the possibility of attaining socially desirable allocations by maximizing behavior of producers and consumers under suitable decentralized mechanisms (in particular, price systems). Furthermore, these are also useful for obtaining some interesting qualitative results regarding the maximin growth paths. Price characterizations of this sort are scarce in the literature on maximin growth. (For some notable exceptions, see the discussion in Section 5). We show (in Theorem 2) that a feasible path is efficient and equitable iff there is a price sequence associated with it, such that (a) at each date, subject to the constraint that the present value of consumption does not exceed the present value of income, "permanent" consumption is maximized at the program; (b) at each date, intertemporal profit is maximized along the program; (c) the present value of capital and exhaustible resource stocks converge to zero. Conditions (b) and (c) have been discussed elsewhere (Mitra [1978], Burmeister and Hammond, [1977]). Our main result is related to (a).

Using this characterization, we show that along an efficient equitable path investment cannot exceed exhaustible resource rents (Propostion 6). We compare this result with the observation of Hartwick [1977, 1978] that in a continuous time version of our model, if investment equals resource rents, for a competitive path, then the path is equitable. We demonstrate that in a discrete-time model, the competitive conditions, intergenerational equity and Hartwick's condition are incompatible, when the production function is strictly concave (Proposition 7). Furthermore, for an efficient equitable path, investment always falls short of resource rents, under strict concavity of the production function (Theorem 3). Finally, we demonstrate the robustness of Hartwick's rule, by proving that it is "asymptotically valid" in our discrete-time framework, in the sense that the excess of resource rents over investment becomes insignificant over time (Proposition 7 and Theorem 4).

In our concluding section, we provide an informal discussion of an alternative route to arrive at the results on Hartwick's rule, given in Section 6, without using (A.4′).

2. THE MODEL

Consider an economy with a technology given by a production function, G from R^3_+ to R_+. The production possibilities consist of capital input, k, exhaustible resource input, r, labor input, z, and current output, $V=G(k, r, z)$, for $(k, r, z)\geq 0$.[2]

Following Solow [1974] and Stiglitz [1974], $G(k, r, z)$ can be interpreted as current output net of depreciation, or simply as current output, assuming no depreciation of capital. A *total output function*, F, can be defined by

$$F(k, r, z) = G(k, r, z) + k \quad \text{for} \quad (k, r, z) \geq 0. \tag{1}$$

The production function, G, is assumed to satisfy:

(A.1) *$G(k, r, z)$ is concave, homogeneous of degree one, and continuous for $(k, r, z)\geq 0$; it is differentiable for $(k, r, z)\gg 0$.*

(A.2) *G is non-decreasing in k, r, and z, for $(k, r, z)\geq 0$; also, $(G_k, G_r, G_z)\gg 0$ for $(k, r, z)\gg 0$.*[3]

For $(k, r, z)\gg 0$, we define the *capital share* (α), *the resource share* (β), and the *labor share* (γ), in current output $V=G(k, r, z)$ by

$$\begin{aligned} &\alpha = (kG_k)/G(k, r, z);\ \beta = (rG_r)/G(k, r, z); \\ &\gamma = (zG_z)/G(k, r, z). \end{aligned} \tag{2}$$

By (A.1), (A.2), it is clear that for $(k, r, z)\gg 0$, α, β, and γ are positive, and ≤ 1, and $(\alpha+\beta+\gamma)=1$. We denote $\inf_{(k,r,z)\gg 0} \beta$ by $\underline{\beta}$.

The available labor force is assumed to be stationary, and positive, and denoted by $\underline{z}$. We will normalize $\underline{z}=1$. In the following, paths are always defined from non-negative capital and resource stocks. Furthermore, along any path, it is understood that, always, $z_t=1$ for $t\geq 0$ (unless explicitly mentioned otherwise).

A *feasible path* from $(\underline{k}, \underline{m})\geq 0$ is a sequence $\langle k, m\rangle=\langle k_t, m_t\rangle$ satisfying

$$\begin{cases} (k_0, m_0) = (\underline{k}, \underline{m}) \\ k_t \geq 0, \quad \text{and} \quad 0 \leq m_{t+1} \leq m_t \quad \text{for} \quad t \geq 0 \\ F(k_t, m_t - m_{t-1}, 1) - k_{t+1} \geq 0 \quad \text{for} \quad t \geq 0. \end{cases} \tag{3}$$

In (3), m_t is to be interpreted as the resource stock at time t. Associated with a feasible path $\langle k, m\rangle$ from $\langle \underline{k}, \underline{m}\rangle$ is a sequence $\langle r, y, c\rangle$ defined by

[2] For any two n-vectors, u and v, $u\geq v$ means $u_i\geq v_i$ for $i=1,\ldots, n$; $u>v$ means $u\geq v$ and $u\neq v$, $u\gg v$ means $u_i>v_i$ for $i=1,\ldots, n$.

[3] $G_k=(\partial G/\partial k)$, $G_r=(\partial G/\partial r)$, $G_z=(\partial G/\partial z)$.

$$(4)\qquad \begin{cases} r_t = m_t - m_{t+1},\ y_{t+1} = F(k_t, r_t, 1) & \text{for} \quad t \geq 0 \\ c_{t+1} = y_{t+1} - k_{t+1} & \text{for} \quad t \geq 0. \end{cases}$$

In (4), y_{t+1} is to be interpreted as the *total output* at time $(t+1)$; c_{t+1} as the *consumption* at time $(t+1)$; and, r_t as the *resource use* at time t. Notice that (3) and (4) imply that $r_t \geq 0$ for $t \geq 0$, and $\sum_{t=0}^{\infty} r_t \leq \underline{m}$. Henceforth, *a path will always refer to a feasible path.* A path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$ is *interior* if $(k_t, r_t) \gg 0$ for $t \geq 0$. It is said to *maintain a positive consumption level* if $\inf_{t\geq 1} c_t > 0$.

A path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$ *dominates* a path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m})$ if $c_t \geq \bar{c}_t$ for $t \geq 1$, and $c_t > \bar{c}_t$ for some t. A path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m})$ is *inefficient* if there is a path from $(\underline{k}, \underline{m})$ which dominates it. It is *efficient* if it is not inefficient.

The production possibilities can be viewed in the "stock version" as given by a technology set $\mathscr{T}$ of input-output pairs in the following way:

$$(5)\qquad \mathscr{T} = \{[(k, m, z), (y, m', 0)]: \quad 0 \leq y \leq F(k, r, z);\ 0 \leq r \leq (m-m');\ (k, r, z, m') \geq 0\}.$$

It is clear that if $\langle k, m\rangle$ is a path from $(\underline{k}, \underline{m})$, then $[(k_t, m_t, z_t), (y_{t+1}, m_{t+1}, 0)] \in \mathscr{T}$ for $t \geq 0$.

A path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$ is called *competitive* if there is a non-null sequence of non-negative prices $\langle p, q, w\rangle = \langle p_t, q_t, w_t\rangle$ such that for $t \geq 0$,

$$(6)\qquad \begin{aligned} & p_{t+1}y_{t+1} + q_{t+1}m_{t+1} - p_t k_t - q_t m_t - w_t z_t \\ & \geq p_{t+1}y + q_{t+1}m' - p_t k - q_t m - w_t z \\ & \text{for all} \quad [(k, m, z), (y, m', 0)] \in \mathscr{T}. \end{aligned}$$

In other words, the *intertemporal profit maximization condition* (6) is satisfied at each date. A competitive path is said to satisfy the *transversality condition* at the price sequence $\langle p, q, w\rangle$ if

$$(7)\qquad \lim_{t\to\infty} (p_t k_t + q_t m_t) = 0.$$

It is said to have *finite consumption value* if

$$(8)\qquad \sum_{t=1}^{\infty} p_t c_t < \infty.$$

A path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m})$ is a *maximin path* if

$$(9)\qquad \inf_{t\geq 1} \bar{c}_t \geq \inf_{t\geq 1} c_t$$

for every path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$. It is a *non-trival maximin path* if it is a maximin path *and* maintains a positive consumption level. It is an *equitable path* if

$$(10)\qquad \bar{c}_t = \bar{c}_{t+1} \quad \text{for} \quad t \geq 1.$$

It is a *non-trivial equitable path* if it is an equitable path *and* maintains a positive consumption level.

Given a non-negative summable sequence $\langle x\rangle=\langle x_t\rangle$ we will write

$$\sigma_t(x)=\sum_{s=t}^{\infty} x_s \quad \text{for} \quad t\geq 0. \tag{11}$$

3. SOME PRELIMINARY PROPERTIES OF EFFICIENT AND EQUITABLE PATHS

In this section, we note some properties of efficient and equitable paths, which will be useful in the analysis of the next two sections. First, we establish that if there is an equitable path, then there is an efficient path which has the same constant consumption as the original path, for $t=2$ onwards (Proposition 1). Second, we prove that if an efficient path has a positive non-decreasing consumption level, then it is necessarily interior (Proposition 2). We also note a useful price-support property of interior efficient paths given in Mitra [1978].

For our purpose, we will assume that capital is essential in production:

$$G(0, r, z) = 0 \quad \textit{for} \quad (r, z) \geq 0. \tag{A.3$'$}$$

We start with an obvious, but useful, result.

LEMMA 1. *Under (A.1) (A.2), (A.3′), if $\langle \bar{k}, \bar{m}\rangle$ is an inefficient path from $(\underline{k}, \underline{m})$, then there is a path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$ such that $c_t\geq \bar{c}_t$ for all $t\geq 1$, and $c_1>\bar{c}_1$.*

Proof: We provide only a sketch. Note, first, that if $\langle k', m'\rangle$ is a path from $(\underline{k}, \underline{m})$ and $c'_s>c$ for some $c\geq 0$, and $s>1$, then clearly we can find a path $\langle k'', m''\rangle$ from $(\underline{k}, \underline{m})$ with $c''_t=c'_t$ for $t\neq s-1$, s, $c''_s=c$ and $c''_{s-1}>c'_{s-1}$. This is because, by (A.3′), $k'_{s-1}>0$. So we can be reducing c'_s to $c''_s=c$, also reduce k'_{s-1} to k''_{s-1} by just enough (by (A.3′) again) so that $k''_s=k'_s$. This makes c''_{s-1} larger than c'_{s-1} by $(k'_{s-1}-k''_{s-1})>0$. This procedure leaves $k'_t=k''_t$ for $t\neq s-1$, and $m'_t=m''_t$ for all t, so $c'_t=c''_t$ for $t\neq s-1$, s.

If $\langle \bar{k}, \bar{m}\rangle$ is an inefficient path from $(\underline{k}, \underline{m})$, then there is a path $\langle k', m'\rangle$ from $(\underline{k}, \underline{m})$ such that $c'_t\geq \bar{c}_t$ for all t, and $c'_s>\bar{c}_s$ for some $s\geq 1$. If $s=1$, then we are done. If $s>1$, then by using the above argument a finite number of times, the result is established. □

LEMMA 2. *Under (A.1), if $\langle k^n, m^n\rangle$ is a sequence of paths from $(\underline{k}, \underline{m})$, then there is a subsequence $\langle k^{n'}, m^{n'}\rangle$ which converges co-ordinatewise to a path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m})$.*

PROOF. Consider the sequence $\langle x_t\rangle$ defined by $x_0=\underline{k}$, $x_{t+1}=F(x_t, \underline{m}, 1)$ for $t\geq 0$. Clearly, for each n, $(0, 0, 0)\leq(k^n_t, y^n_t, c^n_t)\leq(x_t, x_t, x_t)$ for $t\geq 1$; $k^n_0=\underline{k}$, $z^n_t=1$, $0\leq m^n_t\leq \underline{m}$ for $t\geq 0$. Hence, for each t, $(k^n_t, m^n_t, z^n_t, y^n_{t+1}, c^n_{t+1})$ is a bounded sequence. By the Cantor Diagonal Process, there is a subsequence of n, say n', such that $(k^{n'}_t, m^{n'}_t, z^{n'}_t, y^{n'}_{t+1}, c^{n'}_{t+1})$ converges to some $(\bar{k}_t, \bar{m}_t, \bar{z}_t,$

$\bar{y}_{t+1}, \bar{c}_{t+1})$ for each t. Using (A.1), (3) and (4), $\langle \bar{k}_t, \bar{m}_t \rangle = \langle \bar{k}, \bar{m} \rangle$ is a path from $(\underline{k}, \underline{m})$. Note that since $m_t^{n'} \to \bar{m}_t$ for $t \geq 0$, so $r_t^{n'} \to \bar{r}_t$ for $t \geq 0$. □

PROPOSITION 1. *Under (A.1), (A.2) and (A.3′), if there is an equitable path* $\langle \hat{k}, \hat{m} \rangle$ *from* $(\underline{k}, \underline{m})$ *with* $\hat{c}_t = d$ *for* $t \geq 1$, *then there is an efficient path* $\langle \bar{k}, \bar{m} \rangle$ *from* $(\underline{k}, \underline{m})$ *with* $\bar{c}_t = d$ *for* $t \geq 2$, *and* $\bar{c}_1 \geq d$.

PROOF. Let $A = [\langle k, m \rangle : \langle k, m \rangle$ is a path from $(\underline{k}, \underline{m})$ and $c_t \geq d$ for $t \geq 2]$. Let $A_1 = [c : c = c_1$ for some $\langle k, m \rangle$ in $A]$

A_1 is non-empty since d belongs to A_1. Also, A_1 is bounded, since for any path $\langle k, m \rangle$ from $(\underline{k}, \underline{m})$, we have $0 \leq c_1 \leq F(\underline{k}, \underline{m}, 1)$. Let $\tilde{c}_1$ be the 1.u.b. of A_1. Then, there is a sequence $\langle k^n, m^n \rangle$ in A such that c_1^n converges to $\tilde{c}_1$. By Lemma 2, there is a path $\langle \bar{k}, \bar{m} \rangle$ from $(\underline{k}, \underline{m})$ with $\bar{c}_1 = \tilde{c}_1$, and $\bar{c}_t \geqq d$ for all $t \geq 2$.

We must have $\bar{c}_t = d$ for $t \geq 2$. Otherwise, using the argument in the proof of Lemma 1, we can find a path $\langle k', m' \rangle$ such that $c'_t \geq d$ for $t \geq 2$, and $c'_1 > \bar{c}_1 = \tilde{c}_1$, a contradiction to $\tilde{c}_1$ being the 1.u.b. of A_1. Using exactly the same argument, $\langle \bar{k}, \bar{m} \rangle$ must also be efficient. □

LEMMA 3. *Under (A.2), if* $\langle k, m \rangle$ *is an efficient path from* (k, m), *with* $c_{t+1} \geq c_t$ *for* $t \geq 1$, *then* $k_{t+1} \geq k_t$ *for* $t \geq 0$.

PROOF. Suppose, on the contrary, there is $\tau \geq 0$, such that $k_{\tau+1} < k_\tau$. Clearly, there is a path $\langle k', m' \rangle$ from $(k_{\tau+2}, m_{\tau+2})$ with $(k'_t, m'_t) = (k_{t+\tau+2}, m_{t+\tau+2})$ for $t \geq 0$ and $c'_t = c_{t+\tau+2}$ for $t \geq 1$. Since $m_\tau \geq m_{\tau+1} \geq m_{\tau+2}$, there is a path $\langle k'', m'' \rangle$ from (k_τ, m_τ) with $(k''_0, m''_0) = (k_\tau, m_\tau)$, $(k''_t, m''_t) = (k'_{t-1}, m'_{t-1})$ for $t \geq 1$ and $c''_1 = G(k_\tau, r_\tau + r_{\tau+1}, 1) + (k_\tau - k_{\tau+2}) \geq G(k_{\tau+1}, r_{\tau+1}, 1) + (k_\tau - k_{\tau+1}) + (k_{\tau+1} - k_{\tau+2}) > G(k_{\tau+1}, r_{\tau+1}, 1) + (k_{\tau+1} - k_{\tau+2}) = c_{\tau+2} \geq c_{\tau+1}$; and $c''_{t+1} = c'_t = c_{t+\tau+2} \geq c_{t+\tau+1}$ for $t \geq 1$. This proves that $\langle k, m \rangle$ is inefficient. This contradiction proves that $k_{t+1} \geq k_t$ for $t \geq 0$. □

For our next result, we strengthen (A.3′) and assume that both capital and the resource are essential in production.

(A.3) $$G(k, 0, z) = 0 = G(0, r, z).$$

PROPOSITION 2. *Under (A.1)–(A.3), if* $\langle k, m \rangle$ *is an efficient path from* $(\underline{k}, \underline{m})$, *with* $c_{t+1} \geq c_t$ *for* $t \geq 1$, *and* $c_1 > 0$, *then* $k_{t+1} > k_t$ *for* $t \geq 0$, *and* $m_{t+1} < m_t$ *for* $t \geq 0$; *furthermore, the path is interior.*

PROOF. By Lemma 3, $k_{t+1} \geq k_t$ for $t \geq 0$. If $m_t = m_{t+1}$ for some $t = \tau$, then $r_\tau = 0$, and by (A.3), $G(k_\tau, r_\tau, 1) = 0$. Since $c_{\tau+1} > 0$, so $k_{\tau+1} < k_\tau$, a contradiction. Since $m_{t+1} \leq m_t$ for $t \geq 0$, so $m_{t+1} < m_t$ for $t \geq 0$.

Clearly, $\underline{k} > 0$; otherwise, if $\underline{k} = 0$, then by (A.3), $c_1 = G(\underline{k}, r_0, 1) + \underline{k} - k_1 = -k_1 > 0$, since $c_1 > 0$, a contradiction. Since $k_{t+1} \geq k_t$ for $t \geq 0$, by Lemma 3, so $k_t > 0$ for $t \geq 0$. Since $m_{t+1} < m_t$ for $t \geq 0$, so $r_t > 0$. Hence, $\langle k, m \rangle$ is interior.

To prove that $k_{t+1} > k_t$ for $t \geq 0$, suppose, on the contrary, that $k_{t+1} = k_t$ for some $t=T$. Then, there is a path $\langle k', m'\rangle$ from (k_T, m_T) with $(k'_0, m'_0)=(k_T, m_T)$ and $(k'_t, m'_t)=(k_{t+T+1}, m_{t+T+1})$ for $t\geq 1$ and $c'_1=G(k_T, r_T+r_{T+1}, 1)+k_T-k_{T+2}=G(k_{T+1}, r_T+r_{T+1}, 1)+k_{T+1}-k_{T+2}>G(k_{T+1}, r_{T+1}, 1)+k_{T+1}-k_{T+2}$ (since $\langle k, m\rangle$ is interior, and (A.2) holds)$=c_{T+2}\geq c_{T+1}$; and $c'_{t+1}=c_{t+T+2}\geq c_{t+T+1}$ for $t\geq 1$. Hence, $\langle k, m\rangle$ is inefficient, a contradiction. □

As a final result of this section, we note that for an interior competitive path, the present-value price of the exhaustible resource is a constant. A proof is given in Mitra [1978, Proposition 3.1 and Theorem 4.1].

LEMMA 4. *Under (A.1) and (A.2), if an interior path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$ is competitive at the price sequence $\langle p, q, w\rangle$, then*

(12) $$q_t = q_{t+1} \quad \text{for} \quad t \geq 0$$

(13) $$F_{r_{t+1}} = F_{r_t} F_{k_{t+1}} \quad \textit{for} \quad t \geq 0.$$

4. EXISTENCE OF AN EFFICIENT EQUITABLE PATH

Our objective in this section is to establish the existence of an efficient equitable path, given arbitrary positive initial capital and resource stocks. In the process, we will notice that such a path is precisely the maximin path from these initial stocks.

Such a result is established by Solow [1974] when the production function, G, is Cobb-Douglas, and the capital share exceeds the resource share. His method is, roughly speaking, to construct a particular type of path, and to show that this path is an efficient equitable path. The construction relies on being able to solve for the capital and resource sequences as functions of time, in closed form. For a general production function like ours, this method does not work. Consequently, we follow an alternative two-step procedure, which can be described in the following way. First, we consider the set of all constant consumption paths, and choose the one with maximum constant consumption. Such a path is shown to exist. Second, we show that this path is efficient.

The difficult step in this procedure is, obviously, the second, for it requires that if the maximum constant consumption path is inefficient — and so, we can improve the lot of *one* generation without worsening the lot of any other — then we can increase the consumption of *every* generation by a *constant* positive amount.

For our existence proof, we need, in addition to (A.1)–(A.3), an assumption which says that the resource is "important" in production (in a sense made precise in (A.4) below). We also need to assume that there exists a non-trivial equitable path from positive initial stocks (see Condition E below). We demonstrate that Condition E is a *necessary* condition for the existence of an efficient equitable program, so its use in our existence proof is clearly justified.

This still leaves one with the question of whether (A.3) and (A.4) are essential for the existence result. Example 1 provides a case of a production function for which (A.3), (A.4) are violated, Condition E holds, and there does not exist an efficient equitable program. We remark that a weaker version of (A.4) (see (A.4′) below) where the resource is "important" *along certain paths*, together with (A.3), are enough to prove the existence theorem. Example 2 shows that (A.4′) is genuinely weaker than (A.4). We do not have an example to show whether (A.4′) is necessary or not. *Whether this could be weakened further or dispensed with remains an open question.*

We start the analysis by proving the existence of a maximin path.

PROPOSITION 3. *Under (A.1) and (A.2) there exists an equitable maximin path from* $(\underline{k}, \underline{m})$.

PROOF. Let $B=\{\langle k, m\rangle : \langle k, m\rangle$ is a path from $(\underline{k}, \underline{m})$, and $c_t=c_{t+1}$ for $t\geq 1\}$. B is non-empty, since the equitable path with zero consumption is in B. Let $B_1=\{c: \langle k, m\rangle$ is in B and $c_1=c\}$. B_1 is bounded since for any path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$, $0\leq c_1\leq F(\underline{k}, \underline{m}, 1\}$. Let c_1^* be the l.u.b. of B_1. Then there exists a sequence $\langle k^n, m^n\rangle$ in B such that c_1^n converges to c_1^*. By Lemma 2, there is a path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m})$ with $\bar{c}_t=\bar{c}_{t+1}$ for $t\geq 1$, and $\bar{c}_1=c_1^*$. We claim that this is a maximin path from $(\underline{k}, \underline{m})$. If not, then there is a path $\langle \tilde{k}, \tilde{m}\rangle$ from $(\underline{k}, \underline{m})$ such that $\inf_{t\geq 1} \tilde{c}_t>\bar{c}_1=c_1^*$. But then, clearly, there exists a path $\langle k', m'\rangle$ from $(\underline{k}, \underline{m})$ with $c_t'=\inf_{t\geq 1} \tilde{c}_t$ for $t\geq 1$, which contradicts the definition of c_1^*. □

It is quite obvious, intuitively, that the existence of an efficient equitable path requires that there be some path which can maintain a positive consumption level. The assumptions (A.1) to (A.3) and (A.4) (see below) are not sufficient to ensure the existence of such a path, as is clear from the studies by Solow [1974], Stiglitz [1974] and Cass and Mitra [1979]. We therefore proceed by considering the following condition.

CONDITION E. There exists a non-trivial equitable path from $(\underline{k}, \underline{m})\gg 0$.

For a complete characterization of production functions, G, for which Condition E is satisfied, the reader is referred to the analysis in Cass and Mitra [1979]. We may now prove

PROPOSITION 4. *Under (A.1)–(A.3) and Condition E, there exists an efficient and equitable path* $\langle \bar{k}, \bar{m}\rangle$ *from some* $(\bar{k}, \bar{m})$ *with* $\bar{c}_t>0$ *for all* $t>0$.

PROOF. By Condition E, there is an equitable path $\langle \hat{k}, \hat{m}\rangle$ from $(\underline{k}, \underline{m})$ with $\hat{c}_t=d>0$ for $t\geq 1$. By Proposition 1, there is an efficient path $\langle k', m'\rangle$ from $(\underline{k}, \underline{m})$ with $c_t'=d$ for $t\geq 2$ and $c_1'\geq d$. Let $(k_1, m_1)=(\bar{k}, \bar{m})$; then the path $\langle \bar{k}, \bar{m}\rangle$ from (k_1', m_1') defined by $(\bar{k}_t, \bar{m}_t)=(k_{t+1}', m_{t+1}')$ for $t\geq 0$ is an efficient and equitable path with $\bar{c}_t=d>0$ for $t\geq 1$. □

For our next result, we need an additional assumption, which conveys (along with (A.3)) that the exhaustible resource is "important" in production.

(A.4) *Given any* $(\tilde{k}, \tilde{r}) \gg 0$, *there is* $\tilde{\eta} > 0$ *such that for all* (k, r) *satisfying* $k \geq \tilde{k}$, $0 < r \leq \tilde{r}$, *we have* $\{[rG_r(k, r, 1)]/G_z(k, r, 1)\} \geq \tilde{\eta}$.

This assumption says, roughly speaking, that the ratio of the share of resource in output to the share of labor in output is bounded away from zero. If the share of resource in output is bounded away from zero $[\beta > 0]$, then clearly (A.4) is satisfied. This stronger assumption has been used by Mitra [1978] to provide a price-characterization of efficient paths.

LEMMA 5. *Under (A.1)–(A.4), if there exists an efficient equitable path* $\langle k, m \rangle$ *from* $(\underline{k}, \underline{m})$, *with* $c_t > 0$ *for all* t, *then given any* $\delta > 1$, *there exists* $\lambda > 1$, *and a path* $\langle k', m' \rangle$ *from* $(\delta\underline{k}, \delta\underline{m})$ *with* $c'_t \geq \lambda c_t$ *for* $t \geq 1$.

PROOF. Since $c_t = d > 0$ for all t, $\underline{k} > 0$ by (A.3). By Proposition 2, $k_{t+1} > k_t$, $0 \leq m_{t+1} < m_t$ for $t \geq 0$ and the path is interior. Hence $k_t \geq \underline{k}$ for $t \geq 0$ and $0 < r_t \leq \underline{m}$ for $t \geq 0$. By (A.4), given $\delta > 1$, there is $\eta > 0$ such that for $k \geq \underline{k}$, $0 < r \leq \delta\underline{m}$,

$$\{[rG_r(k, r, 1)]/G_z(k, r, 1)\} \geq \eta. \tag{14}$$

Choose $\delta > \lambda > 1$, such that

$$[(\lambda - 1)/(\delta - \lambda)] < [\eta/\delta]. \tag{15}$$

Clearly, this can be done. Then, for $t \geq 0$,

$$G(\lambda k_t, \delta r_t, 1) - G(\lambda k_t, \lambda r_t, \lambda) \geq 0. \tag{16}$$

To see this, we write, for $t \geq 0$,

$$\begin{aligned} &G(\lambda k_t, \delta r_t, 1) - G(\lambda k_t, \lambda r_t, \lambda) \\ &\quad \geq G_r(\lambda k_t, \delta r_t, 1)(\delta - \lambda) r_t - G_z(\lambda k_t, \delta r_t, 1)(\lambda - 1) \\ &\quad = (\lambda - 1) G_z(\lambda k_t, \delta r_t, 1)\left[\left\{\frac{(\delta - \lambda) G_r(\lambda k_t, \delta r_t, 1)\delta r_t}{\delta(\lambda - 1) G_z(\lambda k_t, \delta r_t, 1)}\right\} - 1\right] \\ &\quad \geq (\lambda - 1) G_z(\lambda k_t, \delta r_t, 1)\left[\left\{\frac{(\delta - \lambda)\eta}{\delta(\lambda - 1)}\right\} - 1\right] > 0, \end{aligned}$$

since $\lambda k_t \geq \underline{k}$, $0 < \delta r_t \leq \delta\underline{m}$ and by using (15). This verifies (16).

Construct a sequence $\langle k', m' \rangle$ in the following way: $k'_0 = \delta\underline{k}$, $k'_t = \lambda k_t$ for $t \geq 1$ and $m'_t = \delta m_t$ for $t \geq 0$. Then

$$\sum_{t=0}^{\infty} r'_t = \delta \sum_{t=0}^{\infty} r_t \leq \delta\underline{m}.$$

Also, for $t \geq 0$, $c'_{t+1} = F(k'_t, r'_t, 1) - k'_{t+1} \geq G(\lambda k_t, \delta r_t, 1) + \lambda k_t - \lambda k_{t+1} \geq G(\lambda k_t, \lambda r_t, \lambda) + \lambda k_t - \lambda k_{t+1}$ [by using (16)] $= \lambda[G(k_t, r_t, 1) + k_t - k_{t+1}] = \lambda c_{t+1}$. Hence, $\langle k', m' \rangle$ is a path from $(\delta\underline{k}, \delta\underline{m})$ and $c'_{t+1} \geq \lambda c_{t+1}$ for $t \geq 0$. □

The following is a version of (A.4) where the assumption is made along certain efficient equitable paths (whose existence from *some* stocks has been established in Proposition 4).

(A.4′) *If* $\langle k, m\rangle$ *is any efficient and equitable path from some* $(\bar{k}, \bar{m})$ *with* $c_t = d > 0$ *for all* $t \geq 1$, *and* $\delta > 1$, *then there is* $\eta > 0$ *such that* $[r \cdot G_r(k, r, 1)]/G_z(k, r, 1) \geq \eta$ *holds for* $k_t \leq k \leq \delta k_t$ *and* $r_t \leq r \leq \delta r_t$ *for every* $t \geq 0$.

Remark 1. From the first part of the proof of Lemma 5 and the fact that $\delta > 1$, it is immediate that (A.4) implies (A.4′).

Remark 2. From the proof of Lemma 5, it is clear that the inequalities (15) and (16) can be established in the same way directly from (A.4′). Given $\delta > 1$, $\eta > 0$ can be chosen as assured by (A.4′); given δ and η, $\lambda > 1$ and $\lambda < \delta$ can be chosen to satisfy (15). Since $\lambda < \delta$, (A.4′) can be used to obtain (16). Lemma 5 therefore holds with (A.4) replaced by (A.4′).

We now state and prove the main result of this section.

THEOREM 1. *Under* (*A*.1)–(*A*.4), *there exists an efficient equitable path from* $(\underline{k}, \underline{m}) \gg 0$ *if and only if Condition E holds.*

PROOF. (*Necessity*) Suppose there exists an efficient equitable path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m}) \gg 0$. If Condition E does not hold, then $\bar{c}_t = 0$ for $t \geq 1$. Since $(\underline{k}, \underline{m}) \gg 0$, this implies that $\langle \bar{k}, \bar{m}\rangle$ is inefficient, a contradiction.

(*Sufficiency*) By Proposition 3 and Condition E, there exists a non-trivial equitable maximin path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$. Suppose this is not efficient, then by Proposition 1, there is an efficient path $\langle \bar{k}, \bar{m}\rangle$ from $(\underline{k}, \underline{m})$ with $\bar{c}_1 > c_1 = d$ and $\bar{c}_t = d$ for $t \geq 2$. Then clearly there are paths $\langle k', m'\rangle$ from $(\bar{k}_1, \bar{m}_1)$ with $(k'_t, r'_t, z'_t, y'_{t+1}, c'_{t+1}) = (\bar{k}_{t+1}, \bar{r}_{t+1}, \bar{z}_{t+1}, \bar{y}_{t+2}, \bar{c}_{t+2})$ for $t \geq 0$ and $\langle \hat{k}, \hat{m}\rangle$ from $(\bar{k}_2, \bar{m}_2)$ with $(\hat{k}_t, \hat{r}_t, \hat{z}_t, \hat{y}_{t+1}, \hat{c}_{t+1}) = (\bar{k}_{t+2}, \bar{r}_{t+2}, \bar{z}_{t+2}, \bar{y}_{t+3}, \bar{c}_{t+3})$ for $t \geq 0$. Clearly, both are efficient and equitable with $\hat{c}_t = c'_t = d > 0$ for all $t \geq 0$. By Proposition 2, therefore, both are interior and $k'_{t+1} > k'_t$ for $t \geq 0$, $\hat{k}_{t+1} > \hat{k}_t$ for $t \geq 0$.

Let $\bar{c}_1 - d = \varepsilon_1 > 0$. Then clearly there is a path $\langle k'', m''\rangle$ from $(\underline{k}, \underline{m})$ such that $c''_1 = d + \varepsilon_1/2$, $k''_1 = \bar{k}_1 + \varepsilon_1/2$, $m''_1 = \bar{m}_1$ and $(k''_t, m''_t) = (\bar{k}_t, \bar{m}_t)$ for $t \geq 2$. Since $\langle k', m'\rangle$ is interior, therefore $0 < r'_0 = \bar{r}_1 = \bar{m}_1 - \bar{m}_2 = m''_1 - m''_2 = r''_1$. Since $k''_1 > \bar{k}_1$, hence by (A.2), $c''_2 = F(k''_1, r''_1, 1) - k''_2 > F(\bar{k}_1, \bar{r}_1, 1) - k''_2 = \bar{c}_2$ (since $k''_2 = \bar{k}_2$) $= d$. Let $\varepsilon_2 = c''_2 - d > 0$. By (A.1) and (A.2), we can find $0 < \theta < 1$ such that $F(k''_1, \theta r''_1, 1) - k''_2 = d + \varepsilon_2/2$. Hence there is a path $\langle \tilde{k}, \tilde{m}\rangle$ from $(\underline{k}, \underline{m})$ such that $\tilde{c}_1 = c''_1 = d + \varepsilon_1/2$, $\tilde{c}_2 = d + \varepsilon_2/4$, $\tilde{k}_1 = k''_1$, $\tilde{k}_2 = k''_2 + \varepsilon_2/4 = \bar{k}_2 + \varepsilon_2/4$, $\tilde{m}_1 = m''_1 = \bar{m}_1$, $\tilde{m}_2 = m''_2 + (1-\theta) r''_1 = \bar{m}_2 + (1-\theta) r''_1$, $(\tilde{k}_t, \tilde{m}_t) = (k''_t, m''_t) = (\bar{k}_t, \bar{m}_t)$ for $t \geq 3$.

It is clear then, that there is $\delta > 1$ such that $\tilde{k}_2 \geq \delta \bar{k}_2$ and $\tilde{m}_2 \geq \delta \bar{m}_2$. Since $\langle \hat{k}, \hat{m}\rangle$ is an efficient and equitable path from $(\bar{k}_2, \bar{m}_2)$ with $\hat{c}_t = d > 0$ for all t, therefore by Lemma 5, there is $\lambda > 1$ and a path $\langle \hat{\hat{k}}, \hat{\hat{m}}\rangle$ from $(\delta \bar{k}_2, \delta \bar{m}_2)$ such that $\hat{\hat{c}}_t \geq \lambda \hat{c}_t = \lambda d$ for all $t \geq 1$. Hence there is a path $\langle k^*, m^*\rangle$ from $(\underline{k}, \underline{m})$ such that $k^*_1 = \tilde{k}_1$, $k^*_2 = \delta \bar{k}_2$, $m^*_1 = \tilde{m}_1$, $m^*_2 = \delta \bar{m}_2$, $(k^*_t, m^*_t) = (\hat{\hat{k}}_{t-2}, \hat{\hat{m}}_{t-2})$ for all $t \geq 3$, $c^*_1 \geq \tilde{c}_1 \geq d + \varepsilon_1/2$, $c^*_2 \geq \tilde{c}_2 \geq d + \varepsilon_2/4$ and $c^*_t \geq \lambda d$ for $t \geq 3$. But then $\langle k^*, m^*\rangle$ is a path from $(\underline{k}, \underline{m})$ with $\inf_t c_t^* > d = \inf_t c_t$ which contradicts $\langle k, m \rangle$ is a maximin path. Hence our supposition that $\langle k, m \rangle$ is inefficient is false. □

Remark 3. It is clear from Remark 2 and the proof of Theorem 1 that we can replace (A.4) in the hypothesis of Theorem 1 by the weaker assumption (A.4′).

We now note a simple corollary which follows directly from Theorem 1.

COROLLARY 1. *Under (A.1)–(A.4), a path $\langle k, m \rangle$ from $(\underline{k}, \underline{m}) \gg 0$ is an efficient equitable path if and only if it is a non-trivial maximin path.*

The following example shows that when (A.3) and (A.4) are violated, an efficient equitable path need not exist.

Example 1. Let $G(k, r, z) = k^{1/4} r^{3/4} + z$; $\underline{k} = \underline{m} = 1$. First, note that Condition E is satisfied. The sequence $\langle \bar{k}, \bar{m} \rangle$, defined by $\bar{k}_t = \underline{k} = 1$, $\bar{m}_t = 1$ for $t \geq 0$, is clearly a path (with $\bar{r}_t = 0$, $\bar{c}_{t+1} = 1$, $\bar{y}_{t+1} = 2$ for $t \geq 0$) and it is a non-trivial equitable path. Next, it can be checked that both (A.3) and (A.4) are violated.

We demonstrate, now, that 1 is the maximum constant consumption level, for any equitable path. Suppose, on the contrary, that there is a path $\langle k, m \rangle$ from $(\underline{k}, \underline{m})$, with $c_t = (1+e)$ for $t \geq 1$, where $e > 0$. Then, for $t \geq 0$, we have $1 + e = c_{t+1} = k_t^{1/4} r_t^{3/4} + 1 + k_t - k_{t+1}$, or $e = k_t^{1/4} r_t^{3/4} + k_t - k_{t+1}$. Define $\langle \hat{k}_t \rangle$ as follows: $\hat{k}_0 = \underline{k}$, $\hat{k}_{t+1} = \hat{k}_t^{1/4} r_t^{3/4} + \hat{k}_t$ for $t \geq 0$. Then, clearly, $k_t \leq \hat{k}_t$ for $t \geq 0$; also $\hat{k}_{t+1} \geq \hat{k}_t$ for $t \geq 0$. Hence, $\hat{k}_{t+1}^{3/4} - \hat{k}_t^{3/4} \leq [\hat{k}_{t+1}/\hat{k}_t^{1/4}] - [\hat{k}_t/\hat{k}_t^{1/4}] = r_t^{3/4}$. Hence, for $T \geq 0$, $[\hat{k}_{T+1}^{3/4} - \hat{k}_0^{3/4}] \leq \sum_{t=0}^{T} r_t^{3/4} \leq [\sum_{t=0}^{T} r_t]^{3/4} [T+1]^{1/4}$ {by Holder's inequality} $\leq [T+1]^{1/4}$, since $\underline{m} = 1$. Hence, there is a scalar $0 < A < \infty$, such that $\hat{k}_t \leq A(t+1)^{1/3}$ for $t \geq 0$. Now, for $t \geq 0$, $k_{t+1} - k_t = k_t^{1/4} r_t^{3/4} - e \leq \hat{k}_t^{1/4} r_t^{3/4} - \text{e}$. Hence, for $T \geq 0$, $k_{T+1} - k_0 \leq [\sum_{t=0}^{T} \hat{k}_t]^{1/4} [\sum_{t=0}^{T} r_t]^{3/4} - (T+1)e$ {by Holder's inequality} $\leq A^{(1/4)} \left\{ \int_1^{T+2} x^{1/3} dx \right\}^{1/4} - (T+1)e \leq (3A/4)^{1/4} [(T+2)^{4/3}]^{1/4} - (T+1)e = (3A/4)^{1/4} (T+2)^{1/3} - (T+1)e$. Then for T large, $k_T < 0$, a contradiction. (The method used here is a discrete-time analog of the technique used by Solow [1974].)

Thus, we have shown $\langle \bar{k}, \bar{m} \rangle$ is a maximin path from $(\underline{k}, \underline{m})$. However, this path is clearly inefficient, since the sequence $\langle k', m' \rangle$ defined by $k'_t = \underline{k} = 1$, $m'_0 = 1$, $m'_t = 0$ for $t \geq 1$ (with $r'_0 = 1$, $r'_t = 0$ for $t \geq 1$; $c'_1 = 2$, $c'_t = 1$ for $t \geq 2$; $y'_1 = 3$, $y'_t = 2$ for $t \geq 2$) is a path from $(\underline{k}, \underline{m})$, which dominates $\langle \bar{k}, \bar{m} \rangle$.

The next example demonstrates that (A.4′) is genuinely weaker than (A.4). Here (A.1)–(A.3) and (A.4′) are satisfied, (A.4) is violated and efficient equitable paths exist from every positive initial stock. The example also demonstrates that the class of production functions for which assumptions (A.1)–(A.3) and (A.4′) are satisfied is wider than the class of Cobb-Douglas production functions.

Example 2. Let $G(k, r, z) = z \cdot \log [1 + (r^\alpha k^\beta)/z^{\alpha+\beta}]$ when $z > 0$, and $G(k, r, 0) = 0$ for every $(k, r) \geq 0$. Here $\beta - 2\alpha \geq 1/2$, $\alpha > 0$ and $\alpha + \beta \leq 1$. It is clear that G is homogeneous of degree one and that G is continuous at (k, r, z) if $z > 0$. It can also be verified that G is continuous at $(k, r, 0)$.

Differentiability of G for $(k, r, z) \gg 0$ is clear. Some routine calculations can be made to verify that G is concave. Assumptions (A.2) and (A.3) can be similarly checked.

To check that Condition E is satisfied, note that $D(k, m)$ (see Section 5) is concave, non-decreasing in (k, m) and $D(0, 0)=0$. Hence, $D(k, m)>0$ for some $(k, m) \gg 0$ verifies that $D(k, m)>0$ for all $(k, m) \gg 0$, i.e., Condition E holds. Let $(\underline{k}, \underline{m})=(4, \sum_{t=1}^{\infty} 1/t^2)$. Let $k_t=4$ for $t=1,\ldots, 3$, $k_t=t$ for $t \geq 3$, $r_t=1/t^2$ for $t \geq 1$. Then $G(k_t, r_t) \geq \log\,[1+t^{\beta-2\alpha}] \geq \log 3 > 1+\varepsilon$ for some $\varepsilon>0$ since $\beta-2\alpha \geq 1/2$. Then $D(\underline{k}, \underline{m}) \geq \varepsilon > 0$.

Next we check that (A.4) is violated. First it can be checked that $[r \cdot G_r(k, r, z)]/G(k, r, z)=[\alpha/(1+\varepsilon)][\varepsilon/\log(1+\varepsilon)]$ where $\varepsilon=r^\alpha k^\beta/z^{\alpha+\beta}$. Then

(a) $\varepsilon \to \infty$ implies that $r \cdot G_r/G(\cdot) \to 0$ and

(b) $0 < a \leq \varepsilon \leq b < \infty$ implies that $0 < \delta_1 \leq [r \cdot G_r/G(\cdot)] \leq \delta_2 < \infty$ for some δ_1, δ_2.

Also, $[z \cdot G_z/G(\cdot)]=1-\{[(\alpha+\beta)\varepsilon]/[(1+\varepsilon)\log(1+\varepsilon)]\}$ hence,

(c) $\varepsilon \to \infty$ implies that $[z \cdot G_z/G(\cdot)] \to 1$ and

(d) $0 < \bar{a} \leq \varepsilon \leq \bar{b} < \infty$ implies that $0 < \bar{\delta}_1 \leq [(z \cdot G_z)/G(\cdot)] \leq \bar{\delta}_2 < \infty$ for some $\bar{\delta}_1, \bar{\delta}_2$.

(a) and (c) show that (A.4) is violated by taking the sequence $(r^t, k^t, z^t)=(1/t, t^2, 1)$. Also (b) and (d) show that $[r \cdot G_r(k, r, z)]/G_z(k, r, z) \geq \eta > 0$ for some η if there are d_1, d_2 such that $0 < d_1 \leq G(k, r, z) \leq d_2 < \infty$.

We now show that (A.4′) is satisfied. If $\langle k, m \rangle$ is an efficient and equitable path, then $G(k_t, r_t, 1) \geq c_t = d > 0$ for all t. Also, by (43) (see Section 7), $k_{t+1}-k_t \leq r_t \cdot Gr_t$ and $r \cdot G_r=[\alpha r^\alpha k^\beta/(1+r^\alpha k^\beta)] \leq \alpha$. Hence, $G(k_t, r_t, 1) \leq \alpha+d$. Let $\delta>1$, then $0 < d \leq G(k_t, r_t, 1) \leq G(k, r, 1) \leq G(\delta k_t, \delta r_t, 1) \leq \delta(\alpha+d)$ for $(k_t, r_t, 1) \leq (k, r, 1) \leq (\delta k_t, \delta r_t, 1)$ for all t. Hence there is $\eta>0$ such that for such $(k, r, 1)$ we have $[r \cdot G_r(k, r, 1)/G_z(k, r, 1)] \geq \eta$. This shows that (A.4′) is satisfied. Hence there exists an efficient and equitable path from every $(\underline{k}, \underline{m}) \gg 0$ by Theorem 1 and Remark 3.

5. PRICE CHARACTERIZATION OF EFFICIENT EQUITABLE PATHS

In this section, we will provide a price characterization of efficient equitable (or maximin) paths, when such paths exist. We will show that a path is efficient and equitable if and only if there is a price sequence such that (a) at each date, subject to the budget constraint that the present-value of consumption does not exceed the present-value of income, "permanent" consumption is maximized at the program; (b) at each date, intertemporal profit is maximized at the program, and (c) the transversality condition is satisfied. This type of characterization is rather prominent in optimal growth theory, but is scarce in the literature on maximin programs.

Conditions (b) and (c) have been discussed in the literature on efficiency (see Mitra [1978]) and maximin programs (see Burmeister and Hammond [1977], and Dixit, Hammond and Hoel [1980]). These characterize *efficient* paths, whether maximin or not (see Mitra [1978]). Clearly, if one *assumes* that a given path is equitable then it is maximin if and only if (b) and (c) hold, which is what Proposition 5 below says. Burmeister and Hammond [1977] show that if (b) and (c) hold, *and the path is equitable*, then it is maximin. Their method of proof, however, is direct, and does not involve showing that the path is efficient; their model is also more general than ours. However, whether these conditions are necessary for a maximin program is an issue they do not address. The point of interest in the present exercise is in characterizing *efficiency and equity* by means of certain value maximization, profit maximization and transversality conditions *alone*. More precisely, without assuming that the given path is equitable we show that conditions (a) through (c) are necessary and sufficient for efficiency and equity (and hence maximin); see Theorem 2 below.

It is clear that compared to the characterization of efficiency or optimality, the condition which is different here (and is the one of main interest) is (a). It is, therefore, worthwhile to try to spell out the meaning of this condition, in somewhat greater detail. If the price-sequence supporting the path $\langle k, m \rangle$ from $(\underline{k}, \underline{m})$ is $\langle p, q, w \rangle$, then the present-value of income at date t, at the path is

$$p_t k_t + q_t m_t + \sum_{s=t}^{\infty} w_s.$$

Consider, now, any capital and resource stock pair $(\underline{k}', \underline{m}')$, and consider any constant consumption level c' which is feasible from the stocks $(\underline{k}', \underline{m}')$. Then, c' can be interpreted as a "permanent" consumption level attainable with these stocks, and the constant input of labor. Condition (a) states, firstly, that the present-value of this "permanent" consumption, c', cannot exceed the present-value of income; that is,

$$\sum_{s=t+1}^{\infty} p_s c' \leq p_t \underline{k}' + q_t \underline{m}' + \sum_{s=t}^{\infty} w_s.$$

Secondly, it states that the present-value of income on the path is exhausted by the present-value of a consumption stream just maintaining the current consumption level; that is,

$$\sum_{s=t+1}^{\infty} p_s c_{t+1} = p_t k_t + q_t m_t + \sum_{s=t}^{\infty} w_s.$$

Thus, if the present-value of income along the path is the budget, and the present-value of any "permanent" consumption stream, c'', is within this budget; that is,

$$\sum_{s=t+1}^{\infty} p_s c'' \leq p_t k_t + q_t m_t + \sum_{s=t}^{\infty} w_s$$

then $c'' \leq c_{t+1}$. In other words, c_{t+1} is the maximum permanent consumption attainable within the present-value budget of $[p_t k_t + q_t m_t + \sum_{s=t}^{\infty} w_s]$.

Together, it means that if the present-value of income of the comparison pair, $(\underline{k}', \underline{m}')$, does not exceed the present-value of income along the path, that is,

$$p_t \underline{k}' + q_t \underline{m}' + \sum_{s=t}^{\infty} w_s \leq p_t k_t + q_t m_t + \sum_{s=t}^{\infty} w_s$$

the "permanent" consumption c' cannot exceed c_{t+1} at the path. This is, as will be clear below, analogous to the price support property of value functions in the optimal growth literature (see McKenzie [1979]).

In Proposition 5 below we essentially state the result on the price characterization of efficiency contained in Theorem 4.1 in Mitra [1978], as applied to efficient equitable paths. Since our assumption (A.4) or (A.4′) are weaker than the assumption employed in Mitra [1978], we need to elaborate the first few steps of the proof. The rest follows from a direct application of the methods used by Mitra [1978]. It may be noted that (A.4) can be employed only along paths for which $\inf k_t > 0$. It was shown in Section 3 that "$\inf k_t > 0$" is satisfied for efficient paths with non-decreasing consumption; but this condition is clearly not satisfied for arbitrary interior efficient paths, which are discussed in Mitra [1978].

PROPOSITION 5. *Under* (*A*.1)–(*A*.4), *a path* $\langle k, m \rangle$ *from* $(\underline{k}, \underline{m}) \gg 0$ *is efficient and equitable iff there is a price sequence* $\langle p, q, w \rangle$ *with* $(p_t, q_t, w_t) \gg 0$ *for* $t \geq 0$, *such that* (17)–(19) *below hold*:

$$\begin{aligned} 0 &= p_{t+1} y_{t+1} + q_{t+1} m_{t+1} - p_t k_t - q_t m_t - w_t z_t \\ &\geq p_{t+1} y + q_{t+1} m' - p_t k - q_t m - w_t z \qquad \text{for all} \\ &\qquad [(k, m, z), (y, m', 0)] \in \mathscr{T},\ t \geq 0. \end{aligned} \tag{17}$$

$$\lim_{t \to \infty} (p_t k_t + q_t m_t) = 0 \tag{18}$$

$$c_t = c_{t+1} \qquad \text{for} \quad t \geq 1. \tag{19}$$

Furthermore, the following inequalities hold along an efficient, equitable path:

$$\sum_{t=0}^{\infty} w_t < \infty \tag{20}$$

$$\sum_{t=1}^{\infty} p_t c_t < \infty. \tag{21}$$

PROOF. (*Sufficiency*) Using (17), (18), we know from Malinvaud [1953, Lemma 5] that $\langle k, m \rangle$ is an efficient path from $(\underline{k}, \underline{m})$, since $p_t > 0$ for $t \geq 0$. By (19), $\langle k, m \rangle$ is equitable.

(*Necessity*) If $\langle k, m \rangle$ is efficient and equitable, since $(\underline{k}, \underline{m}) \gg 0$, therefore, $c_t = d > 0$, for $t \geq 1$. Hence, by Proposition 2, $\langle k, m \rangle$ is interior and $k_{t+1} \geq k_t \geq \underline{k}$ for $t \geq 0$.

Define a sequence $\langle p, q, w\rangle$ as follows:

$$(22)\qquad \begin{aligned} &p_0 = (F_{k_0}/F_{r_0}),\ p_{t+1} = (1/F_{r_t}) &&\text{for} \quad t \geq 0 \\ &w_t = p_{t+1}F_{z_t},\ q_t = 1 &&\text{for} \quad t \geq 0 \end{aligned}$$

Then, by Propositions 3.1 and 3.2 in Mitra [1978], (17) is satisfied at the sequence $\langle p, q, w\rangle$ defined by (22). Also, clearly, $(p_t, q_t, w_t) \gg 0$ for $t \geq 0$. By (A.4), there is $\eta > 0$ such that for $k \geq \underline{k}$, $0 < r \leq \underline{m}$,

$$(23)\qquad \{[rG_r(k, r, 1)]/G_z(k, r, 1)\} \geq \eta.$$

Then for $t \geq 0$, since $k_t \geq \underline{k}$ and $0 < r_t \leq \underline{m}$, by (23),

$$(24)\qquad w_t = [G_z(k_t, r_t, 1)/G_r(k_t, r_t, 1)] \leq (1/\eta)r_t.$$

Since $\sum_{t=0}^{\infty} r_t \leq \underline{m}$, so $\sum_{t=0}^{\infty} w_t < \infty$. Thus (20) is verified.

Now, following exactly the necessity proof of Theorem 4.1 in Mitra [1978], (18) and (21) hold. □

To proceed further, we define a set $\bar{D}(k, m)$, for every pair of stocks $(k, m) \gg 0$. $\bar{D}(k, m) = \{c\colon \langle \bar{k}, \bar{m}\rangle$ is an equitable path from (k, m) and $\bar{c}_1 = c\}$. $\bar{D}(k, m)$ gives the set of consumption levels which can be maintained from the stocks (k, m). Let $D(k, m) = \text{Max}\{c\colon c \in \bar{D}(k, m)\}$. We now state and prove the main result of this section.

THEOREM 2. *Under* (*A*.1)–(*A*.4), *a path* $\langle k, m\rangle$ *from* $(\underline{k}, \underline{m}) \gg 0$ *is efficient and equitable iff there is a price sequence* $\langle p, q, w\rangle$ *with* $(p_t, q_t, w_t) \gg 0$ *for* $t \geq 0$, *and* $\langle p_t\rangle$, $\langle w_t\rangle$ *summable, such that*

$$(25)\qquad \begin{aligned} 0 &= \sigma_{t+1}(p)c_{t+1} - p_t k_t - q_t m_t - \sigma_t(w) \\ &\geq \sigma_{t+1}(p)c - p_t k - q_t m - \sigma_t(w), \qquad \textit{for} \quad c \in \bar{D}(k, m),\ t \geq 0 \end{aligned}$$

and (17), (18) *hold.*

PROOF. (*Sufficiency*) By (17), (18), $\langle k, m\rangle$ is an efficient path from $(\underline{k}, \underline{m})$, by Malinvaud [1953, Lemma 5].

By (17), we have for $s \geq 0$, $p_{s+1}c_{s+1} = p_{s+1}y_{s+1} - p_{s+1}k_{s+1} = (q_s m_s - q_{s+1}m_{s+1}) + (p_s k_s - p_{s+1}k_{s+1}) + w_s$. So for $t \geq 0$, and $T \geq t$,

$$(26)\qquad \sum_{s=t}^{T} p_{s+1}c_{s+1} = [q_t m_t - q_{T+1}m_{T+1}] + [p_t k_t - p_{T+1}k_{T+1}] + \sum_{s=t}^{T} w_s.$$

Since $\langle w_t\rangle$ is summable, so $\sum_{s=t}^{\infty} w_s$ is convergent, and, by (26), so is $\sum_{s=t}^{\infty} p_{s+1}c_{s+1}$. Then, using (18), we have $\sum_{s=t}^{\infty} p_{s+1}c_{s+1} = q_t m_t + p_t k_t + \sigma_t(w)$. By (25), we also have $\sum_{s=t}^{\infty} p_{s+1}c_{t+1} = q_t m_t + p_t k_t + \sigma_t(w)$. So, for $t \geq 0$,

$$(27)\qquad \sum_{s=t}^{\infty} p_{s+1}c_{s+1} = \sum_{s=t}^{\infty} p_{s+1}c_{t+1}.$$

We will now show that $c_{t+1}=c_{t+2}$ for $t\geq 0$. Note that $\sum_{s=t}^{\infty} p_{s+1}c_{s+1}=p_{t+1}c_{t+1}+\sum_{s=t+1}^{\infty} p_{s+1}c_{s+1}=\sum_{s=t+1}^{\infty} p_{s+1}c_{t+2}+p_{t+1}c_{t+1}$, by using (27). Also $\sum_{s=t}^{\infty} p_{s+1}c_{s+1}=\sum_{s=t}^{\infty} p_{s+1}c_{t+1}$ [by (27)] $=\sum_{s=t+1}^{\infty} p_{s+1}c_{t+1}+p_{t+1}c_{t+1}$. So, $c_{t+1}\sigma_{t+1}(p)=c_{t+2}\sigma_{t+1}(p)$. Since $p_t>0$, so $\sigma_{t+1}(p)>0$, and since $\langle p_t\rangle$ is summable, so $\sigma_{t+1}(p)<\infty$. Hence, $c_{t+1}=c_{t+2}$ for $t\geq 0$. This means that $\langle k, m\rangle$ is an equitable path.

(*Necessity*) Since $\langle k, m\rangle$ is efficient and equitable and $(\underline{k}, \underline{m})\gg 0$, hence $c_t=d>0$ for $t\geq 1$. By Proposition 5, (17), (18) are satisfied at a price sequence $\langle p, q, w\rangle$ defined by (22). Furthermore, $(p_t, q_t, w_t)\gg 0$ for $t\geq 0$, w_t is summable by (20), and since $\langle k, m\rangle$ is equitable with $c_t=d>0$ for $t\geq 1$, so by (21), $\langle p_t\rangle$ is also summable.

By (17), (18), we obtain [by the method used in the sufficiency part] $\sum_{s=t}^{\infty} p_{s+1}\cdot c_{s+1}=q_t m_t+p_t k_t+\sigma_t(w)$. Since c_{s+1} is constant for $s\geq 0$, so

(28) $$\sigma_{t+1}(p)c_{t+1} = q_t m_t + p_t k_t + \sigma_t(w) \qquad \text{for} \quad t\geq 0.$$

For any path $\langle k', m'\rangle$ from $(k, m)\geq 0$, we have, by (17), for any $u\geq 0$, and $t=s+u$, (where $s\geq 0$), $p_{t+1}c'_{s+1}=p_{t+1}(y'_{s+1}-k'_{s+1})\leq(q_t m'_s-q_{t+1}m'_{s+1})+(p_t k'_s-p_{t+1}k'_{s+1})+w_t$. Since $\langle w_t\rangle$ is summable, so $\sum_{s=0}^{\infty} p_{t+1}c'_{s+1}$ is convergent. So, we have $\sum_{s=0}^{\infty} p_{t+1}c_{s+1}\leq q_u m+p_u k+\sigma_u(w)$. Now, consider any c in $\bar{D}(k, m)$. Associated with c is an equitable path $\langle k'', m''\rangle$ from (k, m), with $c''_t=c$ for $t\geq 1$. Then, we have for any $u\geq 0$,

(29) $$\sigma_{u+1}(p)c \leq q_u m + p_u k + \sigma_u(w).$$

Clearly, (28) and (29) establish (25). □

Remark 4. It is clear from the proofs that in Proposition 5 and Theorem 2 we can replace (A.4) by (A.4′). The same is true of all the results in Section 6.

6. INVESTMENT, RESOURCE RENTS AND HARTWICK'S RULE

In this section, we compare some of the consequences of our price characterization result (Theorem 2) with the observation of Hartwick [1977, 1978] that if along an interior competitive path, investment equals exhaustible resource rents, then this ensures that the path is equitable. This result is established by Hartwick in a continuous-time version of the model examined here.

We find that, in our discrete-time framework, the price characterization theorem yields the result that investment cannot exceed resource rents for efficient equitable paths [Proposition 6]. Furthermore, if the production function, G, is strictly concave, the competitive conditions, equity and Hartwick's rule are inconsistent for interior paths [Proposition 7]. In fact, along an *efficient* equitable path, investment is strictly smaller than resource rents in each period [Theorem 3].

It should be mentioned here that the difference between our results and

Hartwick's result is not caused by a "time-phasing" problem: the same difference continues to obtain if, for example, we assume that current output is a function of the current period's inputs, rather than of the previous period's inputs, as we have done throughout this paper. Rather, the difference arises because of the continuous and discrete treatment of time.

But if the difference is caused by the distinct treatments of time, one would expect the difference to vanish asymptotically. This is precisely what happens. We show that for efficient equitable paths the ratio of investment to resource rents converges to unity as $t \to \infty$ [Proposition 8]. Furthermore, the difference between investment and resource rents converges to zero as $t \to \infty$, if the sum of the shares of capital and labor is bounded away from zero [Theorem 4]. In other words, Hartwick's rule is true asymptotically, even in a discrete-time framework.

In order to proceed with our analysis, we associate with a path $\langle k, m\rangle$ from $(\underline{k}, \underline{m})$ an *investment sequence* $\langle I\rangle = \langle I_t\rangle$ given by

(30) $$I_{t+1} = k_{t+1} - k_t \qquad \text{for} \quad t \geq 0.$$

PROPOSITION 6. *Under (A.1)–(A.4), if $\langle k, m\rangle$ is an efficient equitable path from $(\underline{k}, \underline{m}) \gg 0$, then*

(31) $$I_{t+1} \leq G_{r_t} r_t \qquad \textit{for} \quad t \geq 0, \quad \textit{and}$$

(32) $$I_{t+1} \geq G_{r_{t-1}} r_t \qquad \textit{for} \quad t \geq 1.$$

PROOF. If $\langle k, m\rangle$ from $(\underline{k}, \underline{m}) \gg 0$ is efficient and equitable, then by Proposition 5, it is competitive at the price sequence $\langle p, q, w\rangle$ defined by (22).

Let $c_t = c_{t+1} = d$ for $t \geq 1$. Then, since $d \in \bar{D}(k_t, m_t)$ for $t \geq 0$, so by using (25) of Theorem 2,

$$\sigma_{t+2}(p)d - p_{t+1}k_{t+1} - q_{t+1}m_{t+1} - \sigma_{t+1}(w)$$
$$\geq \sigma_{t+2}(p)d - p_{t+1}k_t - q_{t+1}m_t - \sigma_{t+1}(w)$$

So $p_{t+1}(k_{t+1} - k_t) \leq q_{t+1}(m_t - m_{t+1}) = q_{t+1}r_t$.[4] Using (22), $(k_{t+1} - k_t) \leq G_{r_t} r_t$, which is (31).

Since $d \in \bar{D}(k_{t+1}, m_{t+1})$ for $t \geq 1$, so by using (25) of Theorem 2,

$$\sigma_{t+1}(p)d - p_t k_t - q_t m_t - \sigma_t(w)$$
$$\geq \sigma_{t+1}(p)d - p_t k_{t+1} - q_t m_{t+1} - \sigma_t(w).$$

[4] It is clear that these inequalities should be valid in more general multi-sector models (where one looks at constant utility rather than constant consumption paths). Interpreting k as a vector of capital stocks and m as a vector of exhaustible resources since $D(k, m)$ is concave, hence at any point (k, m) there is a price support $(1, p, q)$, i.e., $D(k, m) - p \cdot k - q \cdot m \geq D(k', m') - p \cdot k' - q \cdot m'$ for any $(k', m') \geq 0$. In our case, the price support at (k_{t+1}, m_{t+1}) is $[1, p_{t+1}/\sigma_{t+2}(p), q_{t+1}/\sigma_{t+2}(p)]$. In the proof we are using this price support property of the value function $D(k, m)$ at each t to obtain these inequalities, which say that present value of investment (at terminal prices)≤ 0, where investment includes both additions to capital equipment as well as decumulations of resource stocks.

So, $p_t(k_{t+1}-k_t)\geq q_t(m_t-m_{t+1})=q_tr_t$.[5] Using (22), $(k_{t+1}-k_t)\geq G_{r_{t-1}}r_t$ which is (32). □

For our next result, we assume

(A.5) *$G(k, r, 1)$ is strictly concave, i.e., given $(k, r)\neq(k', r')$, and $0<\theta<1$, $G[\theta k+(1-\theta)k', \theta r+(1-\theta)r', 1]>\theta G(k, r, 1)+(1-\theta)G(k', r', 1)$.*

LEMMA 6. *Under (A.1)–(A.3) and (A.5), if $\langle k, m\rangle$ is an interior, competitive, equitable path from $(\underline{k}, \underline{m})\gg 0$, then for $t\geq 0$,*

$$I_{t+1} = G_{r_t}r_t \quad \textit{implies} \quad I_{t+2} > G_{r_{t+1}}r_{t+1} \tag{33}$$

$$I_{t+2} = G_{r_t}r_{t+1} \quad \textit{implies} \quad I_{t+3} < G_{r_{t+1}}r_{t+2} \tag{34}$$

PROOF. If for some $t\geq 0$, $I_{t+1}=G_{r_t}r_t$, then since $\langle k, m\rangle$ is interior, $G_{r_t}r_t>0$, and so $I_{t+1}>0$, i.e., $k_{t+1}>k_t$. Now, $k_{t+2}-k_{t+1}=[F(k_{t+1}, r_{t+1}, 1)-c_{t+2}]-[F(k_t, r_t, 1)-c_{t+1}]=F(k_{t+1}, r_{t+1}, 1)-F(k_t, r_t, 1)$ {since $\langle k, m\rangle$ is equitable}$>F_{k_{t+1}}(k_{t+1}-k_t)+F_{r_{t+1}}(r_{t+1}-r_t)$ {by using $k_{t+1}\neq k_t$ and (A.5)}$=F_{k_{t+1}}G_{r_t}r_t+F_{r_{t+1}}(r_{t+1}-r_t)$ {since $I_{t+1}=G_{r_t}r_t$}$=G_{r_{t+1}}(r_t+r_{t+1})-G_{r_{t+1}}r_t$ {by using (13), since $\langle k, m\rangle$ is interior and competitive}$=G_{r_{t+1}}r_{t+1}$. This proves (33). The proof of (34) is similar, and is therefore omitted. □

Remark 5. Note that the proof of Lemma 6 actually establishes a stronger result than (33), viz., if for some $t\geq 0$, $I_{t+1}\geq G_{r_t}r_t$, then $I_{t+2}>G_{r_{t+1}}r_{t+1}$. This means, clearly, that $I_{t+1}=G_{r_t}r_t$ can hold for at most a single period. Given this observation, the following proposition is self-evident, and is therefore presented without a proof.

PROPOSITION 7. *Under (A.1)–(A.3) and (A.5), if $\langle k, m\rangle$ is an interior competitive, equitable path from $(\underline{k}, \underline{m})\gg 0$, then $I_{t+1}=G_{r_t}r_t$ can hold for, at most, one period.*

THEOREM 3. *Under (A.1)–(A.5), if $\langle k, m\rangle$ is an efficient equitable path from $(\underline{k}, \underline{m})\gg 0$, then*

$$I_{t+1} < G_{r_t}r_t \quad \textit{for} \quad t\geq 0, \quad \textit{and} \tag{35}$$

$$I_{t+1} > G_{r_{t-1}}r_t \quad \textit{for} \quad t\geq 1 \tag{36}$$

PROOF. By Proposition 5, $\langle k, m\rangle$ is competitive.

If (35) is violated for some period, τ, then, by Proposition 6, $I_{\tau+1}=G_{r_\tau}r_\tau$. By Lemma 6, $I_{\tau+2}>G_{r_{\tau+1}}r_{\tau+1}$, which contradicts (31). Similarly, (32) and (34) yield (36). □

[5] The same is true here as in footnote 4, except that the inequality says present value of investment (at initial prices)≥ 0. These rules would be discrete time analogs of one in continuous time models, which say that present value of net investment$=0$ along an efficient equitable path (see Dixit, Hammond and Hoel [1980]).

Remark 6. Theorem 3 shows that an efficient equitable path violates Hartwick's rule in every period, if (A.1)–(A.5) hold.

We now proceed to show that Hartwick's rule is valid "asymptotically."

LEMMA 7. *Under* (*A*.1)–(*A*.3), *if* $\langle k, m\rangle$ *is an efficient path from* (k, m) *with* $c_{t+1} \geq c_t$ *for* $t \geq 1$, *and* $c_1 > 0$, *then* $G_{k_t} \to 0$ *as* $t \to \infty$.

PROOF. By Proposition 2, $\langle k, m\rangle$ is interior, and $I_{t+1} > 0$ for $t \geq 0$. Hence, $G(k_t, r_t, 1) > c_{t+1} \geq c_1$ for $t \geq 0$. Since $r_t \to 0$ as $t \to \infty$, so $k_t \to \infty$ as $t \to \infty$, by (A.1) and (A.3).

If, contrary to the lemma, there is $\theta > 0$, such that $G_{k_t} \geq \theta$ for a subsequence of periods, then $G(k_t, r_t, 1) \geq G_{k_t} k_t \to \infty$ for this subsequence of periods. For this subsequence, $1 = [G(k_t, r_t, 1)/G(k_t, r_t, 1)] = G[\{k_t/G(k_t, r_t, 1)\}, \{r_t/G(k_t, r_t, 1)\}, \{1/G(k_t, r_t, 1)\}]$, by (A.1). Since $G(k_t, r_t, 1) \to \infty$ for the subsequence, so $\{r_t/G(k_t, r_t, 1)\}$ and $\{1/G(k_t, r_t, 1)\} \to 0$ along the subsequence. Then, by (A.3), we must have $\{k_t/G(k_t, r_t, 1)\} \to \infty$ for the subsequence. But then $\{G_{k_t} k_t/G(k_t, r_t, 1)\} \to \infty$ for the subsequence. But by (A.1), we have $[G_{k_t} k_t/G(k_t, r_t, 1)] \leq 1$ for $t \geq 0$, a contradiction. □

PROPOSITION 8. *Under* (*A*.1)–(*A*.4) *if* $\langle k, m\rangle$ *is an efficient equitable path from* $(\underline{k}, \underline{m}) \gg 0$, *then*

$$[I_{t+1}/G_{r_t} r_t] \longrightarrow 1 \quad \textit{as} \quad t \longrightarrow \infty, \quad \textit{and} \tag{37}$$

$$[I_{t+1}/G_{r_{t-1}} r_t] \longrightarrow 1 \quad \textit{as} \quad t \longrightarrow \infty. \tag{38}$$

PROOF. By Proposition 2, $\langle k, m\rangle$ is interior; hence, the magnitudes in (37) and (38) are well defined.

By Proposition 6, for $t \geq 1$, $G_{r_{t-1}} r_t \leq I_{t+1} \leq G_{r_t} r_t$. So, for $t \geq 1$,

$$[G_{r_{t-1}} r_t/G_{r_t} r_t] \leq [I_{t+1}/G_{r_t} r_t] \leq 1. \tag{39}$$

Now, $[G_{r_{t-1}} r_t/G_{r_t} r_t] = [G_{r_{t-1}}/G_{r_t}]$. Since $\langle k, m\rangle$ is efficient and equitable, so it is competitive, by Proposition 5. Since it is interior, so (13) holds. Hence, for $t \geq 1$, $[G_{r_{t-1}}/G_{r_t}] = [1/\{1 + G_{k_t}\}]$. By Lemma 7, $G_{k_t} \to 0$ as $t \to \infty$. Using these facts, $[G_{r_{t-1}} r_t/G_{r_t} r_t] \to 1$ as $t \to \infty$. Hence, by (39), (37) holds. Similar reasoning verifies (38). □

For our final result we need

$$\inf_{(k,r,z) \gg 0} (\alpha + \gamma) > 0. \tag{A.6}$$

THEOREM 4. *Under* (*A*.1)–(*A*,4) *and* (*A*.6), *if* $\langle k, m\rangle$ *is an efficient equitable path from* $(\underline{k}, \underline{m}) \gg 0$, *then*

$$\sup_{t \geq 0} G_{r_t} r_t < \infty \tag{40}$$

$$[I_{t+1} - G_{r_t} r_t] \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow \infty \tag{41}$$

(42) $$[I_{t+1} - G_{r_{t-1}} r_t] \longrightarrow 0 \quad as \quad t \longrightarrow \infty^6.$$

PROOF. By Proposition 2, $\langle k, m\rangle$ is interior; hence, the magnitudes in (40), (41) and (42) are well defined.

By (A.6), there is $0<\theta<1$, such that $rG_r \leq \theta G(k, r, 1)$ for all $(k, r) \gg 0$. By Proposition 6, (31) holds. Hence, for $t \geq 0$, $I_{t+1} \leq G_{r_t} r_t \leq \theta G(k_t, r_t, 1) \leq \theta(I_{t+1}+c_1)$ since $c_t = c_1$ for $t \geq 1$. Hence we have $(1-\theta)I_{t+1} \leq \theta c_1$, or, $I_{t+1} \leq [\theta/(1-\theta)]c_1$. Hence, for $t \geq 0$, $G(k_t, r_t, 1) = c_1 + I_{t+1} \leq c_1 + [\theta/(1-\theta)]c_1 = [1/(1-\theta)]c_1$. Hence, for $t \geq 0$, $G_{r_t} r_t \leq \theta G(k_t, r_t, 1) \leq [\theta/(1-\theta)]c_1$, which proves (40). Now (41) and (42) follow directly by using (37) and (38). □

7. SOME CONCLUDING OBSERVATIONS

This section is devoted to an informal discussion of a possible route to obtain the results on Hartwick's rule presented in Section 6, without making use of the price-characterization theorem and assumption (A.4) or (A.4′). This approach focuses on the isoquants of the value function $D(k, m)$, and its relation to the isoquants of the production function $F(k, r, 1)$.

Suppose from (k_0, m_0) there is a sequence of stocks $\langle k_t, m_t\rangle$ giving a feasible and efficient stream of consumption $c_t = c > 0$ for $t \geq 1$. Then from $(k', m') = (k', m_0 + r' - r_0)$, where $F(k', r', 1) = F(k_0, r_0, 1)$, we can sustain $c_t \geq c$ for all t, since out of the gross output, $F(k', r', 1)$, we can consume c and leave k_1 and m_1, as on the original path, for future use. The set of (k', m') so defined, which is the isoquant of $F(k, r, 1)$ passing through (k_0, r_0) translated by $(0, m_0 - r_0)$ lies above the isoquant of $D(k, m)$ passing through (k_0, m_0). [By the isoquant of $D(k, m)$, we mean the lower boundary of the set of (k, m) with $D(k, m) \geq c$.]

Since both curves are convex, hence $D(k, m)$ has differentiable isoquants with slope $= -[F_r(k_0, r_0, 1)/F_k(k_0, r_0, 1)]$. Since along the given efficient equitable path (k_t, m_t) are points on this isoquant of $D(k, m)$, the slope of the chord $= -[(k_{t+1}-k_t)/(m_t - m_{t+1})]$ is larger (smaller) than the slope of the $D(k, m)$ isoquant at $[k_{t+1}, m_{t+1}]([k_t, m_t])$. This means

(43) $$G_{r_t} = \frac{F_{r_{t+1}}}{F_{k_{t+1}}} \geq \frac{(k_{t+1}-k_t)}{(m_t - m_{t+1})} = \frac{(k_{t+1}-k_t)}{r_t} \geq \frac{F_{r_t}}{F_{k_t}} = G_{r_{t-1}}.$$

Furthermore, if G is strictly concave in (k, r), then a convex combination of two feasible paths will yield *more* than the convex combination of their respective consumption in each period. Hence, the isoquant of $D(k, m)$ passing through (k_0, m_0) will be *strictly* concave, and hence the weak inequalities in (43) will be

[6] It may be noted that this is quite different from saying that the present value of investment (in capital) is asymptotically the same as the value of disinvestment of resources, i.e., $p_t I_t - q_t(m_t - m_{t+1}) \rightarrow 0$ as $t \rightarrow \infty$ (see also footnotes 4 and 5). This will be trivially true in our case since by Proposition 5, $p_t k_t + q_t m_t \rightarrow 0$ along non-trivial efficient equitable paths. We are demonstrating the stronger proposition that investment in physical units is asympototically the same as the resource rents, neither of which is zero in the limit.

replaced by strict inequalities. Now, clearly, the results of Section 6 depend only on obtaining (43), or its strict inequality analogue. Hence, all the results of that section can be obtained without the price characterization result, and hence, without using (A.4) or (A.4′). It may be noted that we were using only competitiveness of non-trivial efficient equitable paths, (not the transversaliy condition), and this follows from interiority (and efficiency) which only requires (A.1)–(A.3) and not (A.4) or (A.4′).

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